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# Dissipative vector fields on the plane with infinitely many attracting hyperbolic singularities

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**Abstract.** For any $b > 0$, there are dissipative analytic vector fields on $\mathbb{R}^2$ which, when restricted to $\mathbb{R} \times (-b, b)$, have positive Jacobian and infinitely many (attracting) singularities.

## 1. Introduction

Let $U$, $V$ and $U \cup V$ be submanifolds of $\mathbb{R}^2$ of dimension two. Let $F: U \cup V \to \mathbb{R}^2$ be a vector field of class $C^1$. We say that $F$ satisfies property $(d, t)$ on $(U, V)$ if

(i) $\det(DF) > 0$ on $U$

(ii) $\operatorname{tr}(DF) < 0$ on $V$

where

$$DF(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)$$

is the Jacobian matrix.

This article is related to the conjecture about global asymptotic stability which claims that if a $C^1$ vector field $F: \mathbb{R}^2 \to \mathbb{R}^2$ has a singularity $p$ and satisfies property $(d, t)$ on $(\mathbb{R}^2, \mathbb{R}^2)$, then the basin of attraction of $p$ is the whole $\mathbb{R}^2$. Here we prove that

**Theorem A.** *If $b > 0$ is a real number and $N = \mathbb{R} \times (-b, b)$, then there is a $C^\omega$ vector field $F = (f, g): \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies property $(d, t)$ on $(N, \mathbb{R}^2)$ (resp. property $(d, t)$ on $(\mathbb{R}^2, N)$) but it has infinitely many singularities, all of them contained in $N$, and therefore attracting hyperbolic.*

We observe that property $(d, t)$ may or may not persist by a change of coordinates. The examples of this work must not be interpreted as indications against the quoted conjecture. Actually we believe that the conjecture is true and we hope that this paper contributes to a better understanding of the problem.

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The conjecture above has been solved affirmatively under additional conditions since Krasovskii's work [**Kra**]. Markus and Yamabe [**M-Y**] considered the case where one of the partial derivatives of $F$ vanishes identically. Hartmann [**Hr1**] solved the problem when $DF(x) + DF(x)^T$ is everywhere negative definite, where $T$ means transposition. Olech [**Ole**] proved the conjecture when there exist constants $\delta > 0$ and $R > 0$ such that $||x|| > R$ implies that $||F(x)|| > \delta$. These Olech's arguments use the condition $\det(DF(x) > 0$ only at the singularities of $F$. The examples of this work show, for instance, that Olech's arguments do not work if the additional assumption is removed. By the work of Meisters and Olech, the conjecture is true for polynomial vector fields [**M-O**]. There is a rich literature on the subject; we suggest the reader [**Mei**] and [**M-O**] for further references and history of the problem. Also Hartman's book [**Hr2**] deals with this question. Concerning very recent and important results we wish to mention the works of Gasull, Llibre and Sotomayor [**GLS**], Gasull and Sotomayor [**G-S**], and Gorni and Zampieri [**G-Z**].

Let $F = (f, g)$ be a vector field as in Theorem A. The arguments used to prove the result are such that the foliations induced by $df$ and $dg$ and the set $f^{-1}(0) \cap g^{-1}(0)$ (of the singularities of the vector field $F$) are very well described. See in fig. 1 the foliation induced by $df$, where the arrows represent –at their starting points– directions of the vector field $\mathrm{grad}(f)$ and the straight line segments represent the connected components of $f^{-1}(0)$. See in fig. 2 the foliations induced by $df$ (with dotted lines) and $dg$, where the small black balls represent points of $f^{-1}(0) \cap g^{-1}(0)$ and the arrows represent directions of $\mathrm{grad}(g)$. It follows that the phase portrait of the vector field $F$ must be as shown in fig. 3.

The Lemma 1 and fig. 1 show the coordinate function $f|_N$ and the foliation induced by $df|_N$, respectively. The rest of Section 2 is devoted to the construction of the function $g|_N$. The leaves of the foliation induced by $dg|_N$ are also integral curves of the vector field $Z$ which is defined immediately after Lemma 1.

## 2. The examples of Theorem A

First, we shall work in the smooth category. We shall use the following notation: $a > 2$ will be a real number, $M = \mathbb{R} \times (-a, -2)$, and the image and the domain of definition of a function, say $H$, will be denoted by $\mathrm{Im}(H)$ and $\mathrm{Dom}(H)$, respectively.

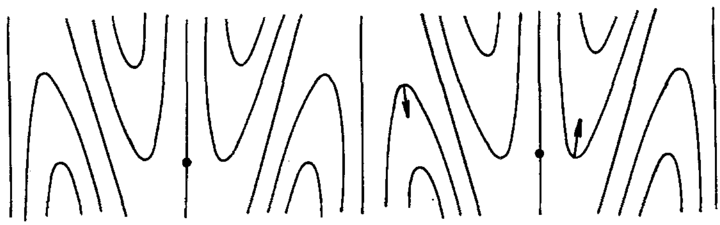

Figure 1

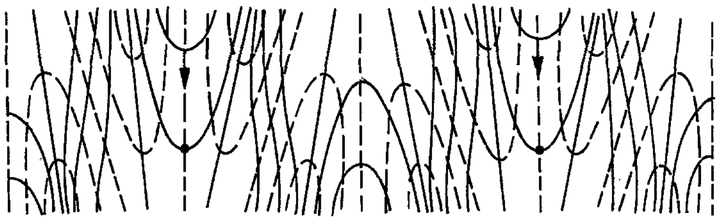

Figure 2

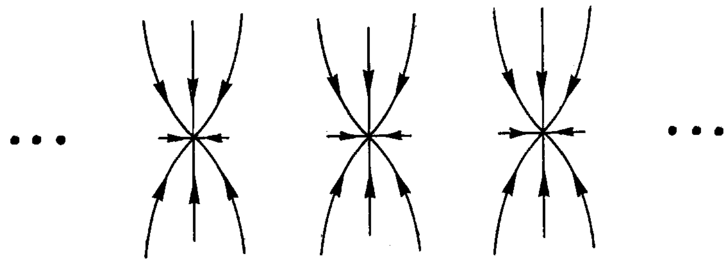

Figure 3

**Lemma 1.** *There exists a smooth submersion* $f: \overline{M} \mapsto \mathbb{R}$ *satisfying the following*

(1.a) *the foliation induced by the 1-form df is that of fig. 1, where the arrows are representing vectors of* $\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$;

(1.b) $f$ *is invariant by a (rigid) translation* $T: \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $T(\overline{M}) = \overline{M}$;

(1.c) *for all* $(u,v) \in M$, $\operatorname{grad} f(u,v) \notin \{(x,y) \in \mathbb{R}^2 / x \geq 0, y \leq \frac{x}{4}\}$;

(1.d) $f^{-1}(0)$ *consists of infinitely many leaves of the foliation induced by* $df$; *and*

(1.e) *for all* $(s,t) \in M$, $10 \geq \|\operatorname{grad} f(s,t)\| \geq 0.1$ *where* $\|(u,v)\| = \sqrt{u^2+v^2}$.

**Proof.** Let $\varphi: \mathbb{R}^2 \mapsto \mathbb{R}$ given by $\varphi(x,y) = -xy(x-y)(x+y) = -x^3y + xy^3$.

We have that grad $\varphi(x, y) = (-3x^2y + y^3, -x^3 + 3xy^2)$. We may observe the following

(1.1) grad $\varphi(x, y) = (0, 0)$ if and only if $(x, y) = (0, 0)$

(1.2) $\varphi^{-1}(0)$ consists of the union of the straight lines $\{x = 0\}$, $\{y = 0\}$, $\{x = y\}$ and $\{x = -y\}$

(1.3) The foliation determined by $d\varphi$ is that of fig. 4. The sectors where $\varphi$ is positive (resp. negative) are indicated by $(+)$ (resp. by $(-)$).

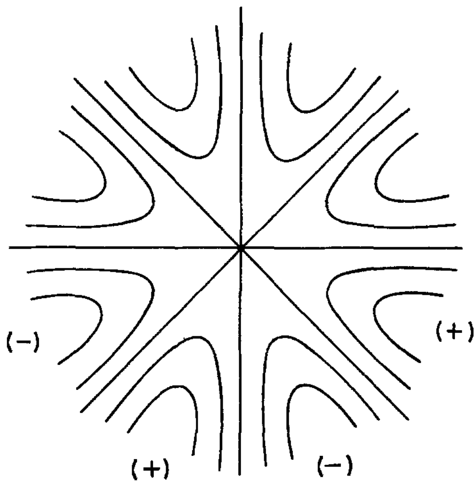


Figure 4

Given $\varepsilon > 0$ small, consider a smooth function $\lambda : \mathbb{R} \mapsto [0, 1]$ such that $\lambda^{-1}(1) = [-\varepsilon, \varepsilon]$, $\lambda^{-1}(0) = (-\infty, -2\varepsilon] \cup [2\varepsilon, \infty)$ and $\lambda'(t) \neq 0$ for all $t \in (-2\varepsilon, -\varepsilon) \cup (\varepsilon, 2\varepsilon)$. Let $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 / y \in [-a, -2]$ and $y \leq x \leq -y\}$. We define the function $\psi : \mathcal{B} \mapsto \mathbb{R}$ by:

$$\psi(x, y) = (x-y)\lambda(x-y) + (x+y)\lambda(x+y) + [1 - \lambda(x-y) - \lambda(x+y)]\varphi(x, y).$$

See in fig. 5 the foliation induced by $d\psi$. There the arrows represent vectors of grad $\psi$.

Observe that $\varphi|_{\mathcal{B}} \geq 0$ nearby $\{x = y\}$ and that $\varphi|_{\mathcal{B}} \leq 0$ nearby $\{x = -y\}$. Therefore, we may easily check that:

(2.1) $\psi^{-1}(0) = \mathcal{B} \cap (\{x = y\} \cup \{x = -y\} \cup \{x = 0\})$;

(2.2) If $(x, y) \in \mathcal{B}$ and $x \leq 0$ (resp. $x \geq 0$), then $\psi(x, y) \geq 0$ (resp. $\psi(x, y) \leq 0$); and

(2.3) There exists a neighborhood of the segment $\{x = y\} \cap \mathcal{B}$ (resp. $\{x =$

$-y\} \cap \mathcal{B})$ such that the level curves of $\psi$, in this neighborhood, are straight line segments parallel to the segment $\{x = y\} \cap \mathcal{B}$ (resp. $\{x = -y\} \cap \mathcal{B}$). Moreover, in this neighborhood, $\operatorname{grad}\psi \equiv (1,-1)$ (resp. $\operatorname{grad}\psi = (1,1)$).

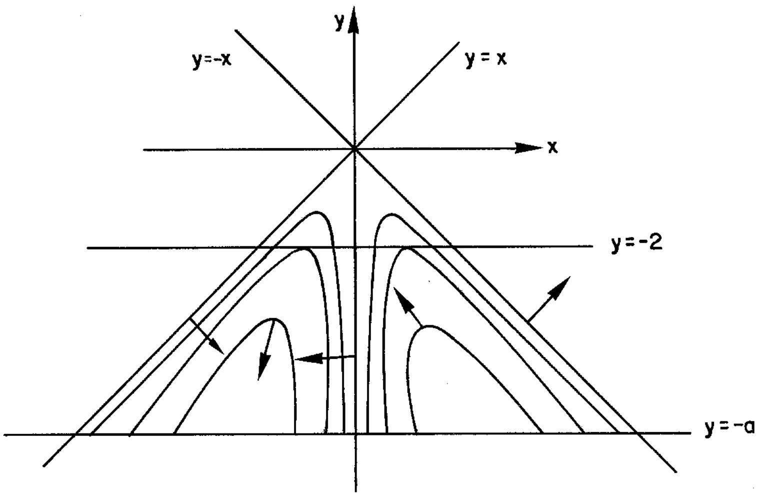


Figure 5

We claim that if $\varepsilon > 0$ is small enough, then there exists $\rho > 0$ such that:

(3) For all $(s,t) \in \mathcal{B}, \operatorname{grad}\psi(s,t) \notin \Gamma = \{x \geq 0$ and $|y| \leq \frac{x}{4}\}$ and $\|\operatorname{grad}\psi(x,t)\| \geq \rho$.

In fact, given $(x,y) \in \mathcal{B}$, we observe that if $0 \leq x-y \leq \varepsilon$ or $0 \geq x+y \geq \varepsilon$ then $\psi(x,y) = x-y$ or $\psi(x,y) = x+y$, respectively. Also, if $(x,y) \in \mathcal{B}$ and $\min\{x-y, -x-y\} \geq \varepsilon$, then $\psi(x,y) = \varphi(x,y)$. therefore, as $\operatorname{grad}\psi$ is continuous, there exist $1 > \rho_1 > 0$ and $\delta > 0$ such that

(4) Let $(x,y) \in \mathcal{B}$. if either $\min\{x-y,-x-y\} \leq \varepsilon+\delta$ or $\min\{x-y,-x-y\} \geq 2\varepsilon$, then $\operatorname{grad}\psi(x,y) \notin \Gamma$ and $1/\rho_1 \geq \|\operatorname{grad}\psi(x,y)\| \geq \rho_1$.

When $(x,y) \in \mathcal{B}$ and $\varepsilon + \delta \leq x - y \leq 2\varepsilon$, the following is satisfied:

(5.1) $\psi(x,y) = (x-y)\lambda(x-y) + (1-\lambda(x-y))\varphi(x,y)$;

(5.2) $\frac{\partial\psi}{\partial x}(x,y) = \alpha(x,y) + (1-\lambda(x-y))\frac{\partial\varphi}{\partial x}(x,y)$

(5.3) $\frac{\partial\psi}{\partial y}(x,y) = -\alpha(x,y) + (1-\lambda(x-y))\frac{\partial\varphi}{\partial y}(x,y)$

where $\alpha(x,y) = (x-y)\lambda'(x-y) + \lambda(x-y) - \lambda'(x-y)\varphi(x,y)$.

Observe that the vector $(\alpha(x,y), -\alpha(x,y)$ is a non-negative constant times $(1,-1)$. Moreover, if $\varepsilon > 0$ is small, the vector $(1-\lambda(x,y))\operatorname{grad}\varphi(x,y)$ points to a direction very close to $(1,-1)$. Also, (5.4) $\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}=(1-\lambda(x,y))[(y-x)(x+(2+\sqrt{3})y)(x+(2-\sqrt{3})y)]$.

Therefore, there exist $\rho_2\in(0,\rho_1)$ such that

(6) If $(x,y)\in\mathcal{B}$ and $\varepsilon+\delta\le x-y\le 2\varepsilon$, then $\operatorname{grad}\psi(x,y)\notin\Gamma$ and $1/\rho_2\ge\|\operatorname{grad}\psi(x,y)\|\ge\rho_2$.

Proceeding as in (4) and (6), we may finish the proof of (3).

Observe that $d\psi$ induces a foliation in $\mathcal{B}$. Therefore, we may consider $\mathcal{B}$ as a foliated "tile" and use copies of it to fill up the band $M=(-\infty,\infty)\times[-a,-2]$ by placing them as in fig. 1. Hence any "tile" $\mathcal{B}'$ of $X$ can be obtained, as the image of $\mathcal{B}$, by an orientation preserving rigid movement of $\mathbb{R}^2$ that leaves invariant $M$. Let $\mathcal{F}$ be the foliation in $M$ induced by the 1-form $d\psi$. Certainly $\mathcal{F}$ is smooth.

We shall extend $\psi\colon\mathcal{B}\mapsto\mathbb{R}$ to a smooth function $f\colon M\to\mathbb{R}$ in the following way. If $\mathcal{B}'=T(\mathcal{B})$ can be obtained as the image of $\mathcal{B}$ by a translation $T\colon\mathbb{R}^2\to\mathbb{R}^2$ that leaves invariant $M$ and $\mathcal{F}$, then we define

$$f|_{\mathcal{B}'}=\psi\circ T^{-1}|_{\mathcal{B}'}.$$

If $\mathcal{B}'=\tilde{T}(\mathcal{B})$ can be obtained as the image of $\mathcal{B}$ by an orientation preserving rigid movement that leaves invariant $M$ and $\mathcal{F}$ but takes the line $\{x=-2\}$ onto the line $\{x=-a\}$, then we define

$$f|_{\mathcal{B}'}=-\psi\circ\tilde{T}^{-1}|_{\mathcal{B}'}$$

Certainly $f$ satisfies the desired conditions of the lemma. □

## The vector field $Z$

By (1.c) of Lemma 1, given $(s,t)\in M$ there exists a unique $\theta=\theta(s,t)\in[\pi/10,2\pi-\pi/10]$ such that

$$(\cos\theta,\sin\theta)=\frac{1}{\|\operatorname{grad}f(s,t)\|}\operatorname{grad}f(s,t).$$

Let

$$Z^{\perp}(s,t)=\left(-\cos\frac{\theta}{2},-\sin\frac{\theta}{2}\right)$$

and

$$Z=\left(\sin\frac{\theta}{2},-\cos\frac{\theta}{2}\right)$$

we shall need of the following.

**Lemma 2.** *Let $\varepsilon \in (0, \pi/2)$ and $K \geq \frac{2}{\sin(\varepsilon/2)}$ be a constant. If $\theta \in [\varepsilon, 2\pi - \varepsilon]$ then:*

(2.a) $\det A \geq 2$

(2.b) $\operatorname{tr} A \leq -1$ *where*

$$A(\theta, K) = A = \begin{pmatrix} \cos\theta & \sin\theta \\ -K\cos\frac{\theta}{2} & -K\sin\frac{\theta}{2} \end{pmatrix}$$

**Proof.** Observe that if $\theta \in [\varepsilon, 2\pi - \varepsilon]$ then $\sin(\frac{\theta}{2}) \geq \sin(\frac{\varepsilon}{2})$. Therefore:

$$\begin{aligned} \det A &= K[-\cos\theta \cdot \sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2}) \cdot \sin\ \theta] \\ &= K[-\{2\cos^2(\frac{\theta}{2}) - 1\}\sin\frac{\theta}{2} + \cos(\frac{\theta}{2}) \cdot 2 \cdot \sin(\frac{\theta}{2}) \cdot \cos(\frac{\theta}{2})] \\ &= K\sin(\frac{\theta}{2}) \geq 2. \end{aligned}$$

This proves (2.a). Now,

$$\begin{aligned} TrA &= \cos\theta - K\sin(\frac{\theta}{2}) \\ &= 2\cos^2(\frac{\theta}{2}) - 1 - K\sin(\frac{\theta}{2}) \\ &\leq 1 - K\sin(\frac{\theta}{2}) \\ &\leq 1 - K\sin(\frac{\varepsilon}{2}) \\ &\leq 1 - 2 = -1. \end{aligned}$$

This proves (2.b). □

**Remark 1.** The following will be used in the proof of Proposition 1. By the way that $f$ has been defined, we may easily see that the vector field $Z$ (defined immediately after the statement of Lemma 1) when restricted to $\mathcal{B}$, has its $y$-coordinate equal to zero only at the segment $\mathcal{B} \cap \{x = 0\}$. □

**Lemma 3.** *Let $\varphi: [-1, 1] \to \mathbb{R}$ be a smooth orientation preserving embedding. Let $s < t$ be points of $[-1, 1]$ and let $k$ and $K > 0$ be real numbers. Given $\delta \in (0, \frac{t-s}{16})$ and $\lambda \in (0, \infty)$, there exists a smooth orientation preserving embedding $\psi = \psi_\lambda: [-1, 1] \to \mathbb{R}$ satisfying:*

(3.a) *for all $u \in [-1, s]$, $\psi(u) = \varphi(u) + k - \varphi(-1)$, in particular, $\psi(-1) = k$;*

(3.b) *for all $u \in [t, 1]$, $\psi(u) = \varphi(u) + \lambda + k - \varphi(-1)$; and*

(3.c) *if* $||\varphi'|| \geq 2k$, *then there exists* $\lambda_0 > 0$ *such that, for all* $\lambda > \lambda_0$,

$$||\psi'_\lambda|| \geq K.$$

**Proof.** Let $\sigma \in \mathbb{R}$ and $\theta: [-1,1] \to [0,1]$ be a smooth function such that $\theta^{-1}(1) = [-1, s] \cup [t, 1]$ and $\theta^{-1}(0) = [s+\delta, t-\delta]$.

Let $\tilde{\psi}_\sigma = \theta\varphi' + \sigma(1-\theta)\varphi'$.

By defining

$$\psi(u) = k + \int_{-1}^{u} \tilde{\psi}_\sigma(v)dv$$

we may easily see that $\psi$ satisfies (3.a) and (3.c) and that $\sigma$ can be chosen so that (3.b) is satisfied. □

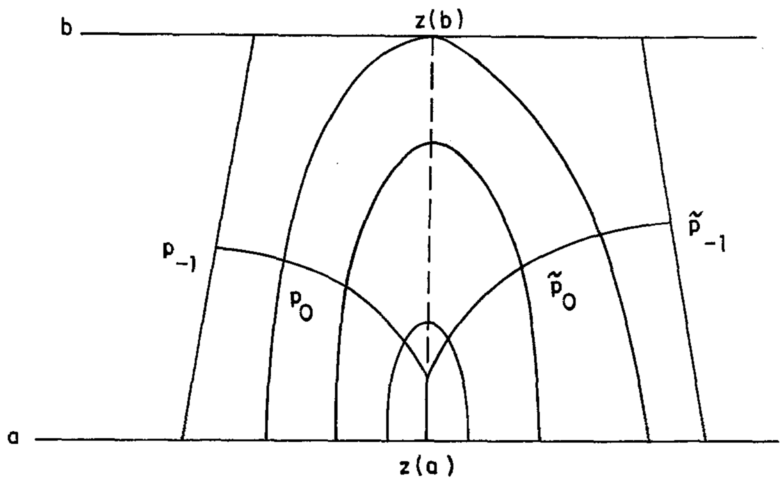


Figure 6

**Lemma 4.** *Let* $a, b, k, K \in \mathbb{R}$ *such that* $a < b$ *and* $K > 0$. *Let* $Q \subset \mathbb{R}^2$ *be a rectangle whose boundary is made up of four smooth embedded curves,* $S \subset \{y = b\}$, $I \subset \{y = a\}$, $L$ *and* $R$. *Suppose that the following is satisfied:*

*There is a smooth vector field* $X$ *on* $Q$, *without singularities, whose phase portrait is as in fig. 6, and which has* $L$ *and* $R$ *as their trajectories. Moreover, there is a smooth map* $z: [a,b] \to Q$ *such that the segment* $Q \cap \{y = c\}$ *is tangent to* $X$ *only at the point* $z(c)$.

*Then, given* $\varepsilon \in \left(0, \frac{b-a}{16}\right)$, *there exists a smooth map* $\Phi: Q(a, b-4\varepsilon) \mapsto \mathbb{R}$ *such that:* $d\Phi(X) \equiv 0$, $\Phi(L \cup R) = k$, *and* $||\text{grad}\,\Phi|| \geq K$. *Here* $Q(a,s)$, *for*

*$s \in [a,b]$, denotes the set*

$$Q(a,s) = Q \cap \{(x,y) \in \mathbb{R}^2 / a \leq y \leq s\}.$$

**Proof.** To simplify matters, given $s \in [a,b]$, the foliation $\mathcal{F}|_{Q(a,s)}$ will be denoted by $\mathcal{F}_s$.

Let $p: [-1,1] \to Q$ be a smooth embedding transversal to $\mathcal{F}$ and such that if $p_t := p(t)$ then

(1) $p_{-1} \in L$, $p_1 = z(a) \in I$, the leaf of $\mathcal{F}$ passing through $p_0$ meets tangentially $S$ and, for $c \in [a,b]$ close to $a$, $z(c) \in \text{Im}(p)$.

Let $P_s$, with $s \in [a,b]$, be the union of the leaves of $\mathcal{F}_s$ that meet $\text{Im}(p)$. By the compactness of $P_b$ and of $\text{Im}(p)$, there exists a smooth map $\varphi: P_b \mapsto \mathbb{R}$, constant along leaves of $\mathcal{F}$, and such that.

(2) $\varphi(p_{-1}) = k$ and $\|\text{grad}\, \varphi\| \geq 2K$.

Let $\tilde{p}: [-1,1] \to Q$ be a smooth embedding transversal to $\mathcal{F}$ and such that if $\tilde{p}_t := \tilde{p}(t)$ then

(3) $\tilde{p}_{-1} \in R$, $\tilde{p}_1 = z(a) \in I$, the leaf of $\mathcal{F}$ passing through $\tilde{p}_0$ is the same as that which passes through $p_0$ and, for $c \in [a,b]$ close to a, $z(c) \in \text{Im}(\tilde{p})$.

There exists a smooth holonomy diffeomorphism, induced by $\mathcal{F}$, between the segments $p((0,1])$ and $\tilde{p}((0,1])$. Thus, by appropriately redefining $p$ and $\tilde{p}$, we may assume that

(4) if $\varepsilon \in (0, \frac{b-a}{16})$ is small enough, then, for all $t \in [\varepsilon, 1]$, $p_t$ and $\tilde{p}_t$ belong to the same leaf of $\mathcal{F}$.

Let $\tilde{P}_s$, with $s \in [a,b]$, be the union of the leaves of $\mathcal{F}_s$ that meet $\text{Im}(\tilde{p})$. By (4) and by the compactness of $\tilde{P}_b$ and of $\text{Im}(\tilde{p})$, there exists a smooth map $\tilde{\varphi}: \tilde{P}_b \mapsto \mathbb{R}$, constant along leaves of $\mathcal{F}$, and such that:

(5) for all $t \in [2\varepsilon, 1], \tilde{\varphi}(\tilde{p}_t) = \varphi(p_t)$, and $\|\text{grad}\, \tilde{\varphi}\| \geq K$.

By the compactness of $\tilde{P}_{2\varepsilon}$ and by Lemma 3 applied to the curves $p$ and $\tilde{p}$, we way redefine $\varphi$ and $\tilde{\varphi}$ in the segments $\text{Im}(p)$ and $\text{Im}(\tilde{p})$, respectively, in such a way that there exists a smooth extension $\Phi: Q(a, b-4\varepsilon) \mapsto \mathbb{R}$, of these newly redefined $\varphi$ and $\tilde{\varphi}$, that satisfies the conditions of this lemma. The parameter $\lambda$, of Lemma 3, makes possible to find these new definitions, of $\varphi$ and $\tilde{\varphi}$, that are compatible. □

**Proposition 1.** *Let $a > 0$ be a real number. There exists a smooth vector field $F = (f,g): \mathbb{R} \times (-a,a) \to \mathbb{R}^2$ which satisfies property $(d,t)$ on $(M,M)$ but it has infinitely many (attracting) singularities.*

**Proof.** As property $(d,t)$ is invariant under translations $x \mapsto x+v$ of $\mathbb{R}^2$, we shall prove that, for $a > 2$, there are smooth vector fields $F = (f,g): M = \mathbb{R} \times (-a,-2) \to \mathbb{R}^2$ as claimed in this proposition.

Let $f: M \to \mathbb{R}$ be the smooth function of Lemma 1, $\mathcal{B} = \{(x,y) \in \mathbb{R}^2 /$ $-a \leq y \leq -2, \quad y \leq x \leq -y\}$, and $\mathcal{F}$ be the foliation induced by the 1-form $df$.

Let $\mathcal{A} \subset \mathcal{B}$ be a rectangle whose boundary consists of four smooth curves: $S \subset \{y = -2\}$, $I \subset \{y = -a\}$, and two integral curves $L \subset f^{-1}(0,\infty)$ and $R \subset f^{-1}(-\infty,0)$ of the vector field $Z$. Observe that $L$ is contained in $\{x < 0\}$ and $R$ is contained in $\{x > 0\}$.

By remark 1, we may use Lemma 4 to conclude that there exists a smooth function $\tilde{g}: \mathcal{A} \mapsto [-1,\infty]$ such that:

$$d\tilde{g}(Z) \equiv 0, \quad \tilde{g}(L \cup R) = -1 \quad \text{and}$$
$$\|\text{grad}\,\tilde{g}\| \geq \frac{20}{\sin(0.1)}. \tag{1}$$

Under these conditions, it follows from (1,c) and (1.e) of Lemma 1 and Lemma 2 that:

(2) $\tilde{F} = (f,\tilde{g}): \mathcal{A} \mapsto \mathbb{R}^2$ satisfies conditions $(d,t)$ on $(\mathcal{A},\mathcal{A})$.

Let $\Gamma_1$ be the set of rectangles $B \subset M$ such that, for some translation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that leaves invariant $\mathcal{F}$ and $M$, we have that $T(\mathcal{A}) = B$.

Let $\Gamma_2$ be the set of rectangles $B \subset M$ such that $T(\mathcal{A}) = B$, where $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid movement that leaves invariant $M$ and $\mathcal{F}$, that preserves orientation, and that takes the line $\{y = -2\}$ onto the line $\{y = -a\}$.

Let $\Gamma_3$ be the set of rectangles $B \subset M$ that are the closure of connected components of $M - \cup\{B/B \in \Gamma_1 \cup \Gamma_2\}$.

We extend $\tilde{g}$ to $\cup\{B/B \in \Gamma_1\}$ in the following way; If $B \in \Gamma_1$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$ is the translation such that $T(B) = \mathcal{A}$, then we define $\tilde{g}: B \to \mathbb{R}^2$ by $\tilde{g}|_B := \tilde{g} \circ T|_B$.

Certainly, by (2), $\tilde{F} = (f,\tilde{g}): \cup\{B/B \in \Gamma_1\} \mapsto \mathbb{R}^2$ satisfies condition $(d,t)$ on $(\text{Dom}(\tilde{F}), \text{Dom}(\tilde{F}))$.

By construction, there is a translation $S: \mathbb{R}^2 \to \mathbb{R}^2$ and there are elements $B_3, B_3' \in \Gamma_3$, $B_2 \in \Gamma_2$ distributed as in fig. 7 and whose union forms a connected set (a rectangle) $B$ which meets $\mathcal{A}$ and $S(\mathcal{A})$

By the same arguments used to prove Lemma 4 and by (1), we may extend $\tilde{g}$ to $B_3 \cup B_3'$ and then to $B_2$ in such a way that the resulting extension, still denoted by $\tilde{g}$, satisfies

(3) $d\tilde{g}(Z) \equiv 0$ and $\|\text{grad}\,\tilde{g}\| \geq \frac{20}{\sin(0.1)}$.

Therefore, by (1,c) and (1,e) of Lemma 1 and Lemma 2,

(4) $\tilde{F} = (f, \tilde{g})|_B : B \to \mathbb{R}^2$ satisfies condition $(d,t)$ on $(\text{Dom}(\tilde{F}), \text{Dom}(\tilde{F}))$.

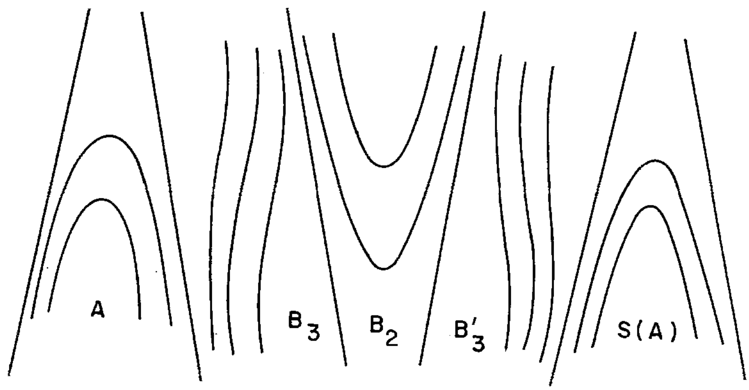


Figure 7

Now, given $p \in M - \cup\{B/B \in \Gamma_1\}$, there exists $n \in \mathbf{N}$ such that $S^n(p)$ belongs to the interior of $B$. We extend $\tilde{g}$ to $p$ by defining $\tilde{g}(p) := \tilde{g} \circ S^n(p)$. The resulting extension, denoted by $\tilde{g}$, is well defined and invariant under $S$. Using (4) we conclude that

(5) $\tilde{F} = (f, \tilde{g}) : M \to \mathbb{R}^2$ satisfies condition $(d,t)$ on $(\text{Dom}(\tilde{F}), \text{Dom}(\tilde{F}))$.

Finally, as $f^{-1}(0)$ meets $\mathcal{B}$, we may find an appropriate constant $c \in \mathbb{R}$ such that if $g = \tilde{g} + c$ then

(6) $F = (f,g) : M \to \mathbb{R}^2$ satisfies not only $(d,t)$ on $(M,M)$ but it has also infinitely many singularities.

This proves the proposition. □

**Proof of Theorem A.** By using a Grauert's approximation result [Gra] we only need to prove the existence of the required vector fields in the smooth category.

Let $\beta : (-b-1, b+1) \to \mathbb{R}$ be a smooth diffeomorphism such that, for all $t \in [-b, b]$, $\beta(t) = t$. Define $H : \mathbb{R} \times (-b-1, b+1) \to \mathbb{R}^2$ by $H(s,t) = ((\beta'(t))^{-1}) \cdot s, \beta(t))$. It is easy to see that

(1.1) $H$ is a diffeomorphism,

(1.2) $\det(DH) \equiv 1$ everywhere, and

(1.3) for all $(s,t) \in \mathbb{R} \times (-b,b)$, $H(s,t) = (s,t)$.

Let suppose that $b > 2$ and let $\tilde{F} : \mathbb{R} \times (-b-1, b+1) \to \mathbb{R}^2$ be a vector field as in Proposition 1. It follows by (1.2) that the vector field $F = H_* \tilde{F}$, that is the pull back of $\tilde{F}$ via $H$, satisfies condition $(d,t)$ on $(N, \mathbb{R}^2)$, and that the vector field $G = \tilde{F} \circ H^{-1}$ satisfies condition $(d,t)$ on $(\mathbb{R}^2, N)$. □

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